# ON THE EQUATION $x^p = [(x^p - [x^p])N]$

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Abstract. We consider equation $x^p = [(x^p - [x^p])N]$ first with $p = 1/3$ and integer $N$, and then generalize results.

## Introduction

In this paper, we study an equation

$$x = [(x^p - [x^p])N], \tag{0.1}$$

where $[x]$ is a floor function, the greatest integer $\leq x$.

First we consider a particular case of $p = \frac{1}{3}$, and integer $N$, and then proceed to generalize to an arbitrary real $p$ and $N$.

## 1. The equation $x = [(\sqrt[3]{x} - [\sqrt[3]{x}])N]$

We first consider the equation

$$x = \left[(\sqrt[3]{x} - [\sqrt[3]{x}])N\right], \tag{1.1}$$

where $[x]$ is the integer part of $x$, and $N$ is a positive integer.

As an example, for $N = 100$, the equation

$$x = \left[(\sqrt[3]{x} - [\sqrt[3]{x}])100\right] \tag{1.2}$$

has one solution $x = 39$, with $\sqrt[3]{x} = 3.39121$.

1.1. **Intermediate equation.** To better understand, how the solutions to (1.1) arise, let's consider an intermediate equation, without the outer square brackets:

$$x = (\sqrt[3]{x} - [\sqrt[3]{x}])N. \tag{1.3}$$

This equation has one solution for every interval $k^3 \leq x < (k+1)^3$, where $(k+1)^3 \leq N$. if $N \neq (k+1)^3$ for some $k$, then the last is a cut-off interval $k^3 \leq x < N$, which may or may not have a solution. The Fig. (1) illustrates this.

The equation

$$x = (\sqrt[3]{x} - [\sqrt[3]{x}])100 \tag{1.4}$$

has solutions

$$x_1 = 1.03125788101082, \quad x_2 = 9.14884844131658, \quad x_3 = 38.9362415949989. \tag{1.5}$$

The last solution of (1.4), $x = x_3 = 38.9362415949989$, is evidently related to the solution of (1.2), $x_0 = 39$. We see that $x_3$ differs from $x_1$ and $x_2$ by being close under an integer $x_0 = 39$. After some consideration, we can understand how it relates to $x_0$ being a solution to (1.2). We define $\Delta x = x_0 - x$, $\Delta y = (a + \frac{x_0}{N}) - (a + \frac{x}{N})$, where

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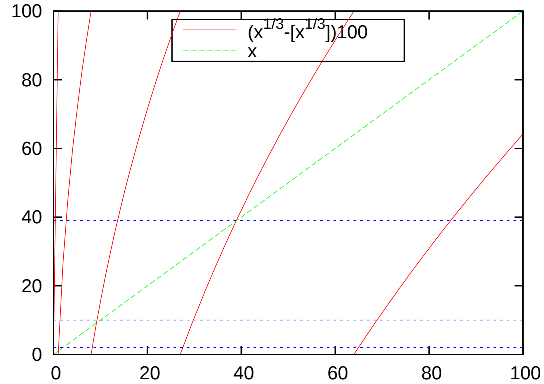


FIGURE 1. (Color online) Solutions of $x = (\sqrt[3]{x} - [\sqrt[3]{x}])\,100$. There are solutions for each interval $(k^3, (k+1)^3)$, $k = 1, 2, 3$. Note closeness to an upper integer of the third root.

$a = [\sqrt[3]{x}] = \sqrt[3]{x} - \frac{x}{N} = \frac{\Delta x}{N}$. In order for $x_0$ to be a solution for (1.2), $x$ must be such, that increasing $x$ by $\Delta x$ will change $\sqrt[3]{x}$ by at least $\Delta y$, but less than by $\Delta y + \frac{1}{N}$:

$$\Delta y \le (\sqrt[3]{x + \Delta x} - \sqrt[3]{x}) < \Delta y + \frac{1}{N}. \tag{1.6}$$

The left inequality of (1.6) is always satisfied. The right inequality may be approximated

$$\frac{\Delta x}{3x^{\frac{2}{3}}} < \frac{\Delta x}{N} + \frac{1}{N}, \tag{1.7}$$

or

$$\Delta x < \frac{1}{\frac{N}{3x^{\frac{2}{3}}} - 1} \tag{1.8}$$

1.2. **Estimate of the density of solutions.** With the condition (1.8), solution happens when lines $x$ and $(\sqrt[3]{x} - [\sqrt[3]{x}])N$ intersect within $\Delta x$ of the next integer number, and $\Delta x$ is the probability of such event. Summing for all approximately $\sqrt[3]{N}$ intersections, we'll get expectation of the number of solutions of the equation (1.1). For large $N$ we can replace sum with integral. We get expected number of solutions

$$n_{sol} = \int_{k=1}^{N^{\frac{1}{3}}} \frac{dk}{\frac{N}{3x^{\frac{2}{3}}} - 1} = \int_{k=1}^{N^{\frac{1}{3}}} \frac{dk}{\frac{N}{3k^2} - 1} \tag{1.9}$$

where $k = \sqrt[3]{x}$. With $N$ increasing to the infinity, the integral converges to 1.

The actual figures. For $0 < N \le 12875$ the numbers of solutions are

0: 4496, 1: 5054, 2: 2480, 3: 691, 4: 130, 5: 21, 6: 2, 7: 1, which gives an average of 9.8888. For $0 < N \le 25408$, the average is 0.99067.

Fig. 2 shows a scatterplot of the actual number of solutions for given $N$.

## 2. GENERALIZATION

2.1. **Generalization of cubic root to a real power.** Now let us consider a general equation (0.1)

$$x = \left[(x^p - [x^p])N\right], \tag{2.1}$$

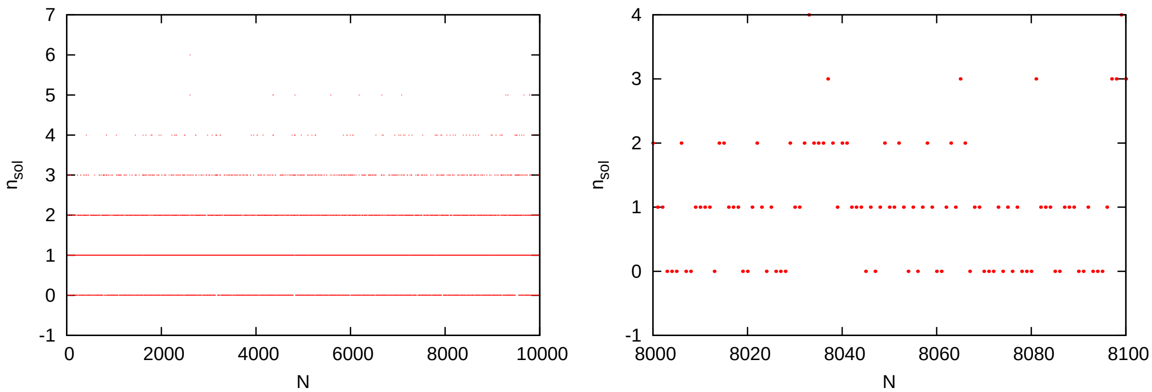


FIGURE 2. (Color online) Number of solutions for a given $N$. Left: Big picture. Right: Close-up.

It is easy to see that our derivation remains the same, if we replace cubic root with an arbitrary real power $p$, $0 < p < 1$. The estimate for $\Delta x$ becomes

$$\Delta x < \frac{1}{\frac{Np}{x^{1-p}} - 1}, \tag{2.2}$$

and estimate of the number of solutions

$$n_{sol} = \int_{k=1}^{N^p} \frac{dk}{\frac{Np}{x^{1-p}} - 1} = \int_{k=1}^{N^p} \frac{dk}{\frac{Np}{k^{\frac{1-p}{p}}} - 1} \tag{2.3}$$

With $N$ increasing to the infinity, the integral also converges to 1.

Computational check for $p = \frac{1}{4}$, $0 < N \leq 15721$, gives an average 0.98753.

2.2. **Generalization of $N$ from integer to a real number.** Here again, most of the derivations of the previous section are applicable to a real $N$, so that the formulas for the density of solutions remain the same. Moreover, changing $N$ to $N + \Delta N$, where $0 < \Delta N < 1$, doesn't change most of the solutions, since changing $(x^p - [x^p])N$ to $(x^p - [x^p])(N + \Delta N)$ changes the expression by about $\frac{p}{2}$ on average, and we have $p < 1$.

**Conclusion.**

## References


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